Summer Project on Condensed Matter Physics - IISc, Bangalore 2023

Lecture 4: SSH Model, Edge States

11th May, 2023

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The energy spectrum of the SSH model was derived in this lecture. In addition to that, the case of a semi-infinite chain of ataoms was explored leading to the existence of edge states.

1 Energy Spectrum in SSH Model

Now going straight back to what we were doing in the last lecture, let us remind ourselves of the eigenvalue problem at hand.

$$\begin{pmatrix} -\mu & -\gamma_1 - \gamma_2 e^{\frac{-ip2a}{\hbar}} \\ -\gamma_1 - \gamma_2 e^{\frac{ip2a}{\hbar}} & -\mu \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E_p \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1}$$

Now, if we take $\mu = 0$,

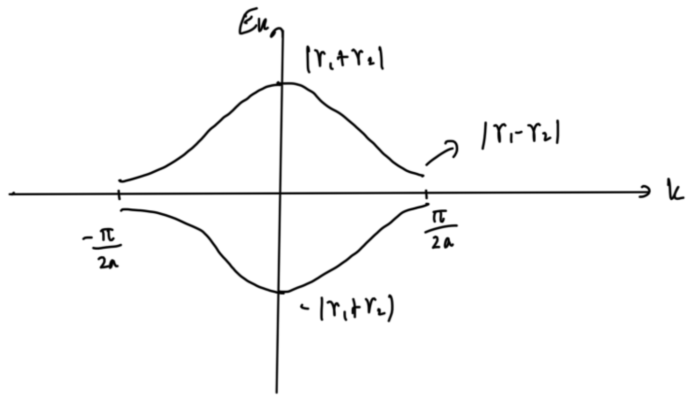
$$\begin{pmatrix} 0 & -\gamma_1 - \gamma_2 e^{\frac{-ip2a}{\hbar}} \\ -\gamma_1 - \gamma_2 e^{\frac{ip2a}{\hbar}} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E_p \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2}$$

$$\implies E_p \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & -\gamma_1 - \gamma_2 e^{-i2ak} \\ -\gamma_1 - \gamma_2 e^{i2ak} & 0 \end{pmatrix} \tag{3}$$

The eigenvalues are:

$$\lambda^2 - (\gamma_1 + \gamma_2 e^{-i2ka})(\gamma_1 + \gamma_2 e^{i2ka} = 0$$

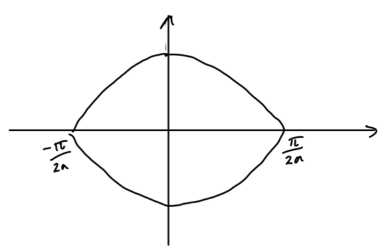
$$\lambda^2 = \gamma_1 + \gamma_2 + 2\gamma_1\gamma_2 cos(2ka)$$
$$E_k = \pm\sqrt{\gamma_1 + \gamma_2 + 2\gamma_1\gamma_2 cos(2ka)}$$



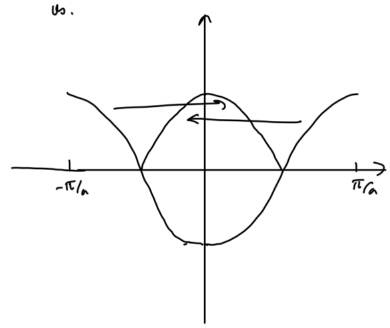
$$\text{Gap} = 2|\gamma_1 - \gamma_2|$$

If we set $\gamma_1 = \gamma_2$, we should get back our previous results for tight-binding Hamiltonian:

$$\begin{aligned} E_k &= \pm\sqrt{2\gamma_1^2 + 2\gamma_1^2 cos(2ka)} \\ &= \pm 2\gamma_1 cos(ka) \end{aligned}$$

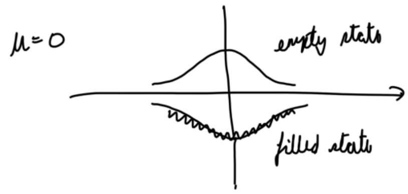


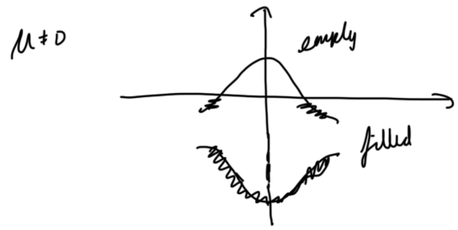
vs.



The case of $\mu \neq 0$:

The μ, if present, will only shift the whole energy spectrum up and down, meaning that it will just add a constant to the energy.





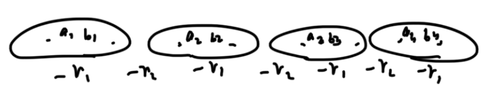
$\rightarrow$Another point to note is that the energy levels do not change under the interchange of γ_1 and γ_2.

What is a semimetal?
When a system has Fermi energy at the edge of a band. Example: graphene.
Interesting facts:
Semimetal: C $\sim T^{\#}$; Metal: C $\sim$ 1 as T; $\rightarrow$ 0 Insulator: C $\sim e^{\frac{\Delta E}{k_B T}}$

Till now the discussion was about an infinite chain of atoms. Let us now look at semi-infinite chain of atoms.

2 Semi-infinite System and Edge Modes

For a semi-infinite chain of atoms in the SSH model, the index of the unit cells will thus be: $n = 1, 2, 3, 4....., \infty$. Then, the following equations become valid for only certain values of j.



$$E\,|a_j\rangle = -\gamma_1\,|b_j\rangle - \gamma_2\,|b_{j-1}\rangle \rightarrow \text{true only for all } j \geq 2 \tag{4}$$
$$E\,|b_j\rangle = -\gamma_1\,|a_j\rangle - \gamma_2\,|a_{j+1}\rangle \rightarrow \text{true for all } j \geq 1 \tag{5}$$

And for $j = 1$,

$$E\,|a_1\rangle = -\gamma_1\,|b_1\rangle \tag{6}$$

Let us look for a solution for which, $|a_j\rangle, |b_j\rangle \to 0$ as $j \to 0$
Take $E = 0$:

$$\begin{aligned} E\,|a_2\rangle &= -\gamma_1\,|b_2\rangle - \gamma_2\,|b_1\rangle \\ \implies 0 &= -\gamma_1\,|b_2\rangle - \gamma_2\,|b_1\rangle \end{aligned}$$

$\therefore |b_1\rangle = 0, |b_2\rangle = 0, ...$

$$\begin{aligned} E\,|b_1\rangle &= -\gamma_1\,|a_1\rangle - \gamma_2\,|a_2\rangle \\ |a_2\rangle &= -\frac{\gamma_1}{\gamma_2}\,|a_1\rangle \end{aligned}$$

Also,

$$\begin{aligned} E\,|b_2\rangle &= -\gamma_1\,|a_2\rangle - \gamma_2\,|a_3\rangle \\ |a_3\rangle &= -\frac{\gamma_1}{\gamma_2}\,|a_2\rangle = \left(-\frac{\gamma_1}{\gamma_2}\right)^2|a_1\rangle \end{aligned}$$

Therefore, in general,

$$|a_{n+1}\rangle = \left(-\frac{\gamma_1}{\gamma_2}\right)^n|a_1\rangle \tag{7}$$

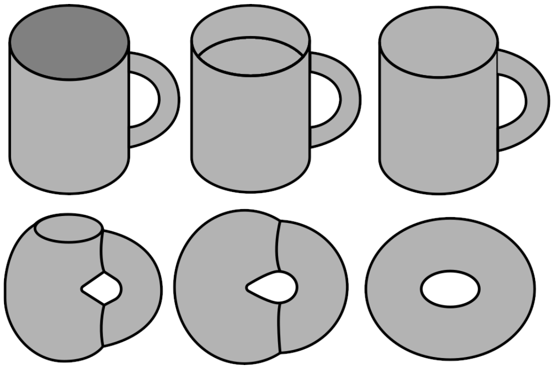
Thus, there is a mode localised at the left and with $E = 0$ if $\left|\frac{\gamma_1}{\gamma_2}\right| < 1$. But what does this mean, or better, how do we understand this property that we have obtained? Let us move on to understand a little topology first.

3 Topology

Topology is a branch of Mathematics which studies the properties of systems which do not change if the system is changed slightly. A very popular saying relating to topology is that a donut and a mug are topologically the same.

→But how are a donut and a mug the same? The answer lies in understanding that both of these items have just one hole and can be transformed into each other by changing the item slightly, without changing the number of holes.

Another example which must be very familiar to people who do a lot of contour integrals is the



number of times a closed curve can go around a certain point. So, if we deform the closed curve without the curve crossing that point, the number does not change. Under such transformations of the curve, the number of times a closed curve goes around a point, called the **winding number**, is called a topological invariant.

$\rightarrow$ Properties associated with such topological invariants are special.
In a 2-D Cartesian plane, a curve can we described in parametric form by $(x(s), y(s))$, where s is a parameter which varies between $0 \leq s \leq 1$. And it forms a closed curve if $x(0) = x(1)$ and $y(0) = y(1)$. Also, we can define an angle ϕ as $\phi = tan^{-1}(y/x)$.
Then we can define the **winding number** as:

$$W = \frac{1}{2\pi} \int_0^1 \frac{d\phi}{ds} \, ds \tag{8}$$

For a circle around centered at origin, $\phi = 2\pi$ and $W = 1$, is the topological invariant.

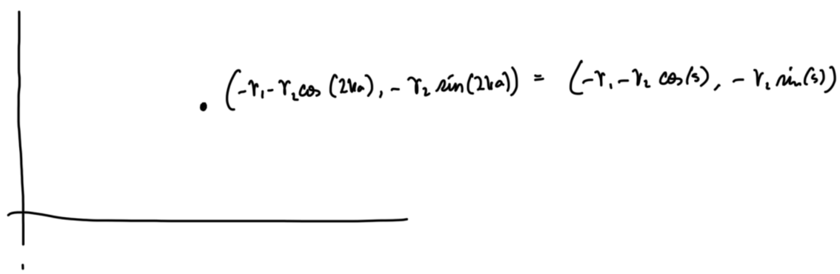
4 Back to the SSH Model

Recall the Hamiltonian with $\mu = 0$,

$$H = \begin{pmatrix} 0 & -\gamma_1 - \gamma_2 e^{-i2ak} \\ -\gamma_1 - \gamma_2 e^{i2ak} & 0 \end{pmatrix} \tag{9}$$

$$= [-\gamma_1 - \gamma_2 cos(2ka)]\sigma^x - \gamma_2 sin(2ka)\sigma^y \tag{10}$$

In the 2-D space of (σ^x, σ^y), this Hamiltonian can be represented as a point:



$(-\gamma_1 - \gamma_2 cos(2ka), -\gamma_2 sin(2ka)) = (-\gamma_1 - \gamma_2 cos(s), -\gamma_2 sin(s))$, defining $2ka = s$ as the parameter.

Now if s is varied from 0 to 2π, we get a closed curve in this space.
$\rightarrow$ What is the winding number of this curve?
Recall that the energy spectrum of the bulk states, now in terms of s, is given by:

$$E_k = \pm\sqrt{(\gamma_1 + \gamma_2 cos(s))^2 + (\gamma_2 sin(s))^2}$$
$$= \sqrt{\gamma_1^2 + \gamma_2^2 + 2\gamma_1\gamma_2 cos(s)}$$

E_k can be zero only if $\gamma_1 = \gamma_2$ and $s = \pi$. In such a case, we cannot calculate the winding number. For $\gamma_1 < \gamma_2$, or better, $\gamma_1 << \gamma_2$: The point in the (σ^x, σ^y) space is $(-\gamma_2 cos(s), -\gamma_2 sin(s)$. This represents a circle centered at the origin and winding number, $W = 1$. Remember that the condition $\gamma_1 << \gamma_2 \implies \left|\frac{\gamma_1}{\gamma_2}\right| < 1$, is actually the condition for the system to have edge states with zero energy. Thus, the winding number in this case is giving us the number of modes at the end of the system.
For $\gamma_1 > \gamma_2$, or $\gamma_1 >> \gamma_2$, the winding number, $W = 0$, and there are no edge states.

For topological systems, given the energy dispersion for the bound states, we can calculate a topological invariant. This topological invariant is found to be equal to the number of modes localised at the ends of an open system.
$\rightarrow$This is called bulk-boundary correspondence.

Also, the length of the system being considered must be $>>$ decay length of the modes localised at the ends (E=0 modes).

$$
\begin{aligned}
|a_{n+1}\rangle &\sim \left(-\frac{\gamma_1}{\gamma_2}\right)^n |a_1\rangle \\
&\sim e^{-\frac{n}{\lambda}} |a_1\rangle \\
\frac{1}{\lambda} &= ln\left(\left|\frac{\gamma_1}{\gamma_2}\right|\right) \\
\lambda &= \frac{1}{ln\left(\left|\frac{\gamma_1}{\gamma_2}\right|\right)}
\end{aligned}
$$

This is the decay length of the edge modes.